**An application of entropy as a risk constraint in Portfolio Optimization.**

By:

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**Abstract**

In this project we pose an optimization problem as described in the framework in [1]. We have used entropy as a measure of risk. As always, the performance of the portfolio has a tradeoff between risk and return. We study the price statistics for a set of stocks for the years of 2009-2011, to first build the empirical distribution, and then compute solution for the optimal portfolio. This solution was tested on actual monthly data for the years 2011-2015.

**Motivation**

The goal of this paper is to obtain an optimal assignment of the amount of the total wealth into each individual asset, where the prices of the assets are known. The difficulty in achieving this in a consistent manner is the probabilistic nature of the valuation of an asset. The analysis in [2] shows a full treatment of random variables, and the quantification of the “randomness” through the use of entropy, represented in a specific logarithmic base. The interpretation of entropy in signal-processing is as the quantification the essential information present in a random signal. The interpretation of entropy in coding theory, is the minimum number of M-ary symbols needed to represent a word sequence, where the entropy is measured in logarithms of base M. The interpretation of entropy in this context could be of similar value, but financial relevance is open to interpretation.

**Literature review**

The first literature to review is the paper by Boyd et. Al . in [1]. This paper provides the framework for this entire project. A condition that has been relaxed from this analysis is the role of transaction costs. In this project, we have ignored the effect of transaction costs, as they are largely going to be assumed to be convex (any solution found with this convex constraint will be global, and will be as good as or worse than the current solution [3]). Another aspect being ignored in this project is the notion of repeating the optimization after fixed intervals. The original problem in [1] is one for making optimal transactions at even intervals where the statistics for that interval have been computed, and a readjustment of the portfolio is warranted. In this project, we focus on the aspect of choosing the best set of assets to buy and hold at the “first” portfolio action, assuming the returns from any previous portfolio activity is zero.

This paper first introduces the maximization problem in standard form. Then it poses the constraints found in real-world situations, such as the constraint of the self-financing portfolio, where the transaction costs of the buys and sells add to zero. It then poses the risk constraint through the use of a variance matrix, and poses a quadratic constraint on the maximization problem. It then proposes an iterative algorithm to find the solution to the maximization problem with the quadratic conditions.

The paper has been critical in providing the foundation for this project. Furthermore, it states the common goal of maximizing the worth of one’s investment, as a mathematical problem, formalizing the connection between investment and optimization theory.

The second literature used as reference in this project is [2], which is a famous text on information theory. A basic introduction to Information theory is presented in this book, including the analysis of multivariate random phenomena. The extended application of this will be seen in the future works section.

**Formulation**

Let us represent a total amount of wealth X, which needs to be fully used for investments. Let there be N assets, and a vector x, where x and represents the percentage of the total wealth X assigned to asset i. There also exists a vector w, where w that represents the average return on each of the stock. We then assume that wealth X allotted must fully be traded and held for the period of analysis.

The extension made in this project is the imposing a condition on the total entropy in the portfolio. Each asset I will have an associated empirical distribution, and associated entropy. The total entropy in the portfolio can at most be. In this project we have used a control parameter α that served as a restraint metric on our entropy bound. Note that [2] gives a reason for us to assume that can serve as the most liberal upper bound constraint, as there is a possibility that the entropy in the portfolio will be less than this, because of mutual information, or statistical dependence, between the assets. Finally, there is an implied condition of the non-negativity constraint on x. This reflects the problem of choosing the best stocks, rather than for making the best trade. The optimization has the following form:

= α

**Solution**

The solution to solve this process was through various stages. First the a set of stocks were picked and the data was downloaded from google’s archives [4]. The data was then cleaned, tabularized, and averaged. The daily growth rates were also calculated. This part was done through python and is the element in the codebase A. Then a low-level utility was developed to generate empirical probability distributions on the stock data, which was done in C++, and is the code in codebase B. The final mathematical portion was made possible using the “fmincon” function available in the standard student version of Matlab.

The solution results where the entropy control factor was set to 1, the portfolio reflected an 88% investment in Google, and 1% in the remaining 11 assets. When the control factor was made small (and hence more severe), was a small investment in Ford Corporation, and nothing else.

**Simulation results**

First we assessed the performance of a portfolio with a natural risk constraint, where the entropy of the bought assets equals the max portfolio entropy. This was then compared to a portfolio whose risk constraint was tightened. The comparison was based on the level return in the portfolio’s worth with time over 50 months, with respect to the starting point. The result is as follows:



Another thing analyzed from the study was the role of the “α” parameter, in the found return rate in the test data. The following is an observation we made that is a result of the statistical activity of the assets during the test period.



The effect of a long range of control parameter values is shown below.



We then chose a good value of the control parameter, analyzed the return in the portfolio at the good value of the parameter. The result is shown below.



**Conclusion**

By precomputing the empirical probability distribution of the valuation of assets, we can provide a linear risk measure for portfolio optimization, without the need for a quadratic constraint as posed in [1]. Naturally, the tradeoff is the distributions will need to be recomputed for a reevaluation of the entropies of the assets. We see that using this measure of risk does not circumvent a tradeoff between the risk and return. Although in the dataset of our testing, the riskier asset happened to show an overall reduction in value. We have seen the effect of controlling the importance of the entropy bound. This method can be used an fast quasi-constraint for more aggressive risk constraints.

**Future Work: Foreign currency exchanges**

The use of entropy as a risk measure can be extended for assets other than stocks. An important subdomain of financial portfolios is the use foreign currency exchange, or forex for short. A problem in forex that is of keen importance is that of arbitrage, which is the utilizing the perceived price discrepancy between currencies in order gain a riskless.

**Definition**

Consider a set of tuples M = [R, C], which is a graph of the form G = (E,V), as described in [5] a tuple set of Edges E and Vertices V. This can be represented as the graph shown below.

The graph shown above will have a starting point S, and destination point D. The original goal of this problem would be to maximize the product of the exchange ratios between for the set of currencies K V , with length D in some ordering f(K), such that f(0) = S and f(L) = D. The mathematical problem can be written as the following problem:

Max

But this problem is NP hard. Also, in it is the hidden implication of choosing an optimal subset of currencies from V.

Another way to express this problem as one of a graph of additive cost would be to compute the costs of the edges of the graph, as return ratios based on the starting currency. Therefore for each starting currency-ending currency tuple, we would have to re-express the weights on the graph in terms of return (if the currency value of two adjacent countries is the same, the new cost between then, express in return rate, will be 0. If the value of a new currency is twice as of the old one, which reflects a 100% increase in value, would equal 1).

We can then change the problem from a maximization problem of finding the longest path (which is NP hard), to one of finding the shortest path. In deterministic cases, we can use the algorithms known in literature currently, to find the shortest distance. This shortest value can be used to determine the optimal arbitrage path. An algorithmic description would be as shown:

**An algorithm for a Systematic Elimination of Currencies for Forex Arbitrage**

* *Fix starting currency, and ending currency*
* *Calculate the input currency holder’s relative return ratios when transacted through each of the other currencies in the market, to make the additive graph with negative weights.*
* *If lowest cost path found = cost of direct transaction between source and destination:*

*Ignore current source currency, as the direct cost is the best you can do*

*Else*

*If lowest cost path found < direct*

*Buy the destination currency through the optimized route, and sell back to the source currency back directly.*

*Else*

*Buy the destination currency directly, and sell back to the source currency back through the optimized route.*

An additional work that can be done in this, would be to find the role of the entropies of the return ratios in the new framework, or the exchange rate between the currencies in the original framework. As the exchange rates as stochastic variables, the analysis of such graphs, and shortest path algorithms found must account for the role of risk in the overall expected return. As can be seen through the work in the rest of this project, I recommend the use of entropy to measure risk.

**References**

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